6.832 Midterm

Name: SOLUTIONS

October 23, 2014

Please do not open the test packet until you are asked to do so.

- You will be given 85 minutes to complete the exam.
- Please write your name on this page, and on any additional pages that are in danger of getting separated.
- We have left workspace in this booklet. Scrap paper is available from the staff. Any scrap paper should be handed in with your exam.
- YOU MUST WRITE ALL OF YOUR ANSWERS IN THIS BOOKLET (not the scrap paper).
- The test is open notes.
- The test is out of 35 points.

Good luck!

Problem	Possible	Your Score
Problem 1	5	
Problem 2	10	
Problem 3	10	
Problem 4	10	
Total	35	

Problem 1 (5 pts) *The Linear Quadratic Regulator*

Suppose you are given a stabilizable linear system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u},$$

and the cost-to-go function

$$J(\mathbf{x}_0) = \int_0^\infty \left[\mathbf{x}(t)^T\mathbf{Q}\mathbf{x}(t) + \mathbf{u}^T(t)\mathbf{R}\mathbf{u}(t)\right] dt, \quad \mathbf{x}(0) = \mathbf{x}_0,$$

with $\mathbf{Q}^T = \mathbf{Q} \succ 0$, $\mathbf{R}^T = \mathbf{R} \succ 0$, *and* $eig(\mathbf{A}) \neq 0$. *I can plot the level-set of the optimal cost-to-go function* $J^*(\mathbf{x}) = 1$*; for this problem formulation the optimal cost-to-go is a positive-definite quadratic function,* $J^* = \mathbf{x}^T\mathbf{S}\mathbf{x}$, $\mathbf{S} = \mathbf{S}^T \succ 0$ *and the level-set is an ellipse. The optimal control is given by* $\mathbf{u}^* = -\mathbf{K}\mathbf{x}$.

a) *If I were to double the value of* $\mathbf{R}$ *(so* $\mathbf{R}_{new} = 2\mathbf{R}$*), what happens to the 1 level-set of the cost-to-go? (Circle one)*

   (i) *It gets bigger (the volume of the ellipse increases)*

   (ii) *It gets smaller (the volume of the ellipse decreases)*

   (iii) *The size does not change (the volume of the ellipse remains constant)*

   Give a mathematical justification for your answer:

   Solution: *The answer is (ii) - it gets smaller. Intuitively, by increasing the instantaneous cost in this way, the cost-to-go can only increase. To see this formally, write the cost function as*

$$J_{new}(\mathbf{x}_0) = \int_0^\infty \left[\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u}\right] dt + \int_0^\infty \mathbf{u}^T\mathbf{R}\mathbf{u}dt$$

   To be very clear, we let us write $J(\mathbf{x}_0, \mathbf{K})$ *for the cost of executing with the controller* $\mathbf{u} = -\mathbf{K}\mathbf{x}$. *We would like to compare the cost-to-go* $J_{new}(\mathbf{x}_0, \mathbf{K}^*_{new})$ *to the original optimal cost-to-go* $J(\mathbf{x}_0, \mathbf{K}^*)$.

   First observe that the optimal control gains for the new problem, $\mathbf{K}^*_{new}$*, cannot produce a lower cost on the original problem, otherwise they would have outperformed the original optimal controller, so*

$$J(\mathbf{x}_0, \mathbf{K}^*_{new}) \geq J(\mathbf{x}_0, \mathbf{K}^*).$$

   Furthermore, since $\mathbf{R}$ *is strictly positive definite, we have*

$$J_{new}(\mathbf{x}_0, \mathbf{K}^*_{new}) \geq J(\mathbf{x}_0, \mathbf{K}^*_{new}).$$

   Note that by giving $\mathbf{Q}$ *strictly positive definite and* $\mathbf{A}$ *non-zero, we assure that the optimal control gains* $\mathbf{K}^*_{new}$ *are non-zero, which gives* $J_{new}(\mathbf{x}_0, \mathbf{K}^*_{new}) > J(\mathbf{x}_0, \mathbf{K}^*_{new})$ *(except for equality at* $\mathbf{x}_0 = 0$*).*

   Since the cost-to-go is quadratic and strictly greater than the original cost-to-go, the 1-level set is strictly smaller.

b) Now suppose that you have doubled both $\mathbf{Q}$ *and* $\mathbf{R}$*, so*

$$\mathbf{Q}_{new} = 2\mathbf{Q}, \quad \mathbf{R}_{new} = 2\mathbf{R}.$$

Which of the following statements about the resulting optimal cost-to-go,

$$J^* = \mathbf{x}^T \mathbf{S}_{new} \mathbf{x},$$

are true (circle all that apply).

(i) $\mathbf{S}_{new} = \frac{1}{4}\mathbf{S}$
(ii) $\mathbf{S}_{new} = \frac{1}{2}\mathbf{S}$
(iii) $\mathbf{S}_{new} = \mathbf{S}$
(iv) $\mathbf{S}_{new} = 2\mathbf{S}$
(v) $\mathbf{S}_{new} = 4\mathbf{S}$

Give a mathematical justification for your answer:

Solution: *The answer is (iv) -* $\mathbf{S_{new}} = \mathbf{2S}$*. Intuitively, we would expect that if we simply double the instantaneous cost then we would also simply double the cost-to-go. Mathematically, we can see that if* $\mathbf{S}$ *is a solution to the algebraic Riccati equation,*

$$0 = Q - SBR^{-1}B^T S + SA + A^T S,$$

then $\mathbf{S}_{new} = 2\mathbf{S}$ *must be a solution to*

$$0 = 2Q - S_{new}B(2R)^{-1}B^T S_{new} + S_{new}A + A^T S_{new}.$$

c) *Continuing with the formulation in part (b), which of the following statements about the resulting optimal controller,*

$$\mathbf{u}^*_{new} = -\mathbf{K}_{new}\mathbf{x},$$

are true (circle all that apply):

(i) $\mathbf{K}_{new} > \mathbf{K}$

(ii) $\mathbf{K}_{new} = \mathbf{K}$

(iii) $\mathbf{K}_{new} < \mathbf{K}$

where the inequality is taken element-wise. Remember the $-\mathbf{K}$ in the description above.

Give a mathematical justification for your answer:

Solution: *The answer is (ii) - $\mathbf{K}_{new} = \mathbf{K}$. Following the solution to part (b), we know that*

$$\mathbf{u}^*_{new} = -(2\mathbf{R})^{-1}\mathbf{B}^T\mathbf{S}_{new}\mathbf{x} = -\mathbf{K}\mathbf{x}.$$

This is as we would expect - doubling the cost doubles the cost-to-go, but does not actually change the optimal policy.

Problem 2 (10 pts) *Lyapunov analysis*

a) *Suppose you have a system* $\dot{\mathbf{x}} = f(\mathbf{x})$ *with* $f(0) = 0$ *and a positive-definite scalar function* $V(\mathbf{x})$ *where you have successfully verified that*

$$\dot{V}(0) = 0$$
$$\dot{V}(\mathbf{x}) < 0, \quad \forall \mathbf{x} \textit{ with } 0 < \sum_i |x_i| \leq 1.$$

Describe the set of initial conditions for which you can guarantee that the system will arrive at the origin as $t \to \infty$*. Explain your answer.*

Solution: *The conditions above guarantee that* V *does not increase while the system is inside the region described (which you may have recognized as the unit ball for the* l_1 *norm). But we can only guarantee convergence for trajectories which* never leave *this region. The best we can guarantee is the sub-level set,* $V \leq \rho$ *where* ρ *is the* smallest *value of* V *on the boundary described by* $\sum_i |x_i| = 1$.

b) Consider an uncertain nonlinear system of the form

$$\dot{\mathbf{x}} = f_1(\mathbf{x}) + \alpha f_2(\mathbf{x}), \quad \alpha = \{0.8, 1.1\}.$$

In words, the uncertain gain α is known to take one of exactly two values – either 0.8 or 1.1, but we do not know apriori which one.

(i) Suppose that you know that the origin, $\mathbf{x} = 0$, is a fixed point for the system $\dot{\mathbf{x}} = f_1(\mathbf{x}) + f_2(\mathbf{x})$. Is the origin guaranteed to be a fixed point for the uncertain system? Circle yes or no.

YES *or* NO

Explain your answer:

Solution: *The answer is NO.*

$$\dot{\mathbf{x}} = 0 \Rightarrow f_1(\mathbf{x}) = -f_2(\mathbf{x}),$$

but does not say anything about $\alpha f_2(\mathbf{x})$.

(ii) Suppose that you are given a radially-unbounded, positive-definite Lyapunov function, $V(\mathbf{x})$, which satisfies the conditions

$$\forall \mathbf{x} \neq 0, \dot{V}(\mathbf{x}, 0.8) < 0, \quad \dot{V}(0, 0.8) = 0,$$
$$\forall \mathbf{x} \neq 0, \dot{V}(\mathbf{x}, 1.1) < 0, \quad \dot{V}(0, 1.1) = 0,$$

where I've used the notation $\dot{V}(\mathbf{x}, \alpha) = \frac{\partial V}{\partial \mathbf{x}} \left[f_1(\mathbf{x}) + \alpha f_2(\mathbf{x}) \right]$.
Which of the following can we conclude about the system

$$\dot{\mathbf{x}} = f_1(\mathbf{x}) + f_2(\mathbf{x})$$

i. The origin is globally stable in the sense of Lyapunov (i.s.L.).
ii. The origin is globally asymptotically stable.
iii. The origin is globally exponentially stable.
iv. None of the above.

Circle all that are true. Provide a mathematical justification for your answer:

Solution: *The first two are true (i and ii). This can be demonstrated by realizing that the original system can be written as an affine combination of the two systems, and therefore we have*

$$\dot{V}(\mathbf{x}, 1) = \frac{1}{3}\dot{V}(\mathbf{x}, .8) + \frac{2}{3}\dot{V}(\mathbf{x}, 1.1) < 0,$$

for $\mathbf{x} \neq 0$ and $\dot{V}(0, 1) = 0$. Since V is radially-unbounded this satifies the conditions to demonstrate global asymptotic stability of the origin, and also stability i.s.L.
Note that this may be surprising given the result in part (a), but in fact knowing that V is strictly decreasing for both systems away from $\mathbf{x} = 0$ implies that $\mathbf{x} = 0$ is the fixed point for the original system as well.

(iii) Suppose that you are given a radially-unbounded, positive-definite Lyapunov function candidate, $V(\mathbf{x})$, which you know fails to satisfy the Lyapunov conditions globally, but you would like use sums-of-squares (SOS) optimization to verify the conditions:

$$\forall \mathbf{x} \in \{\mathbf{x} : V(\mathbf{x}) \leq 1\}, \dot{V}(\mathbf{x}, 0.8) \leq 0$$
$$\forall \mathbf{x} \in \{\mathbf{x} : V(\mathbf{x}) \leq 1\}, \dot{V}(\mathbf{x}, 1.1) \leq 0.$$

Write down a sums-of-squares program by listing the decision variables and all *of the required sums-of-squares constraints in the boxes below.*

find

$$\lambda_1(\mathbf{x}), \lambda_2(\mathbf{x})$$

parameterized by $\lambda_1(\mathbf{x}) = a^T m_1(\mathbf{x})$ and $\lambda_2(\mathbf{x}) = b^T m_2(\mathbf{x})$, where a and b are vectors of decision variables and $m_1(\mathbf{x})$ and $m_2(\mathbf{x})$ are monomial vectors.

(list decision variables)

subject to

$$-\dot{V}(\mathbf{x}, 0.8) + \lambda_1(\mathbf{x})(V(\mathbf{x}) - 1),$$
$$\lambda_1(\mathbf{x}),$$
$$-\dot{V}(\mathbf{x}, 1.1) + \lambda_2(\mathbf{x})(V(\mathbf{x}) - 1),$$
$$\lambda_2(\mathbf{x})$$

is SOS

Note that it is also acceptable to use the same multipliers for both systems, this would potentially be more conservative but would also use less decision variables so could allow you to search over more monomial vector coefficients.

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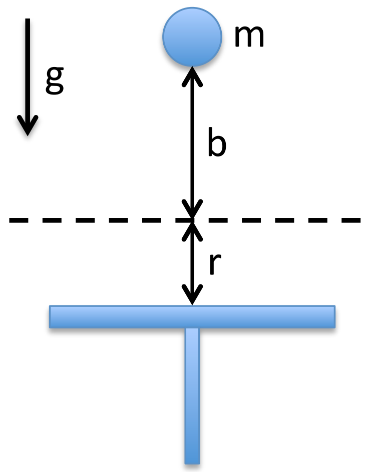
Problem 3 (10 pts) *Trajectory optimization*

Imagine the simple model of a juggling robot illustrated below, which consists of a paddle that can move vertically (with configuration described by the position r, which is negative in the figure) and a ball (modeled as a point with mass m at vertical height b).



Our input is direct control of the velocity of the paddle, resulting in a hybrid state space model with

$$\mathbf{x} = \begin{bmatrix} r \\ b \\ \dot{b} \end{bmatrix}, \quad \dot{\mathbf{x}} = f(\mathbf{x}, u) = \begin{bmatrix} u \\ \dot{b} \\ -g \end{bmatrix},$$

and a simple elastic collision model,

$$\dot{b}^+ = \dot{r} - .9(\dot{b}^- - \dot{r}).$$

We assume the paddle is sufficiently massive to be unaffected by the collision with the ball.

Let us formulate a direct transcription trajectory optimization problem with 11 *knot points to find a periodic solution for this system with exactly one collision per period. We will use the decision variables*

$$\mathbf{x}_0, ... \mathbf{x}_{10}, \quad u_0, ..., u_9, \quad h.$$

and add the dynamic constraints

$$\mathbf{x}_{n+1} = \mathbf{x}_n + f(\mathbf{x}_n, u_n)h,$$

where $h > 0.01$ is the timestep between knot points.

a) *Let us enforce that the paddle and the ball only come into contact at height 0. Write down all of the required constraints that you must add to the program relating to the hybrid guard and reset (aka collision). Your answers should be written in terms of differentiable functions of the decision variables named above.*

Solution: *We can accomplish this by, e.g., requiring that the ball always stays above vertical and the paddle always stays below the ball (except at the endpoints):*

$$b_0 = b_{10} = 0$$
$$r_0 = r_{10} = 0$$
$$r_1 < b_1, r_2 < b_2, ..., r_9 < b_9$$

and we add the periodicity/reset constraint

$$\dot{b}_0 = u_0 - .9(\dot{b}_{10} - u_0).$$

Note that $b_1, ..., b_9 > 0$ *will be enforced automatically given the dynamic constraints (but it does no harm to include it).*

b) *To further constrain the solution, let us require that the apex of the ball occurs at a height of 1. Write any additional constraints required to enforce this. Your answers should be written in terms of differentiable functions of the decision variables named above.*

Solution: *While there are many creative ways to enforce this constraint, the simplest by far is to realize that the solution ball dynamics during the aerial phase will always be symmetric in time (it is the ballistic trajectory of a point mass). Therefore, it is sufficient to add the constraint*

$$b_5 = 1.$$

Note that $\dot{b}_5 = 0$ *will be enforced automatically by the dynamic constraints (but it does no harm to include it).*

c) *Do you expect the problem you have formulated so far to have a unique solution? If not, explain why and provide a reasonable additive-cost objective function written in terms of the decision variables.*

Solution: *The problem so far has done nothing to describe the desired trajectory for the paddle, except to constrain the velocity at time* 0 *(and therefore* 10*) and say that we'd like the paddle to always be below the ball. Therefore, we can express our preference for the motion of the paddle through the objective function. A simple integral cost penalizing paddle velocity should do the trick:*

$$\sum_{i=0}^{9} u_i^2.$$

I would expect the result to be a paddle trajectory the matches the required velocities at the endpoints, and then evenly distributes over the remaining trajectory the velocity required to reset the paddle for the next hit.

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Problem 4 (10 pts) *Nonlinear Dynamics and the Hamilton-Jacobi-Bellman Equations*

a) *Suppose you are given the system*

$$\dot{x} = x + u,$$

where x and u are scalars, and the cost function,

$$J(x_0) = \int_0^\infty g(x(t), u(t))dt, \quad g(x,u) = x^2 + u^2, \quad x(0) = x_0,$$

which we would like to minimize. Suppose you are also given the candidate cost-to-go function

$$\hat{J}(x) = x^4 - x^2 + \frac{1}{4}.$$

Use the optimality conditions to derive the control law associated with this cost-to-go:

$$\hat{u} = \operatorname*{argmin}_u \left[g(x,u) + \frac{\partial \hat{J}}{\partial x} f(x,u) \right].$$

Show your work.

Solution: *Because the argmin is taken over a positive quadratic form in u, we can do the minimization by finding the u for which the gradient of the right-hand side (RHS) is zero:*

$$\hat{u} = \operatorname*{argmin}_u \left[x^2 + u^2 + (4x^3 - 2x)(x+u) \right]$$
$$\frac{\partial RHS}{\partial u} = 2u + (4x^3 - 2x) = 0$$
$$\hat{u} = -2x^3 + x$$

b) *Is the controller derived above optimal? Circle one of the following:*

YES *or* NO *or* INCONCLUSIVE

Solution: *This was a tricky question. The most important point by far was to demonstrate that you understand the fact that the HJB condition is a sufficient condition, but not a necessary one. Substituting $\hat{J}$ and the $\hat{u}$ derived in part (a) into the HJB you will see that it does not match with equality. This would lead one to circle "Inconclusive". Circling "No" using only this justification is absolutely incorrect.*

But it turns out that we do know more for this problem. The dynamics are linear and the objective is a positive quadratic – so the problem formulation is LQR! Since LQR results in a linear optimal feedback and quadratic cost-to-go, we can definitively conclude that the correct solution is "NO"*, the controller is not optimal.*

c) *Using the system and controller from part (a), what are the fixed points of the closed-loop system. For each fixed point, say if it is locally unstable, locally stable i.s.L, locally asymptotically stable, and/or locally exponentially stable.*

Solution: *Using this controller, the closed-loop dynamics are*

$$\dot{x} = 2x - 2x^3.$$

This is just -2 times the cubic example used in the lecture notes. The fixed points are at $x = -1$,0, and 1. Sketching the one-dimesional flow, we can verify graphically that $x = -1$ and $x = 1$ are locally stable fixed points, and that $x = 0$ is locally unstable. Both stable fixed points are stable i.s.L, asymptotically, and exponentially, which can be demonstrated graphically by drawing a line between the $\dot{x}$-curve and the x-axis to represent an exponentially stable linear system which this system converges faster than.

Alternatively, we can verify the stability by evaluating the derivative at the fixed points, and noting that locally, the stable fixed points look like

$$\dot{\bar{x}} \approx (2 - 6(1^2))\bar{x} = -4\bar{x}.$$

d) *Now consider a similar problem, but we will remove the input term from the cost function and add input limits to the system, so that we have*

$$\dot{x} = x + u, \quad |u| \leq 1,$$

and the cost function,

$$J(x_0) = \int_0^\infty g(x(t), u(t))dt, \quad g(x, u) = x^2, \quad x(0) = x_0,$$

which we would like to minimize. Using the same candidate cost-to-go function

$$\hat{J}(x) = x^4 - x^2 + \frac{1}{4},$$

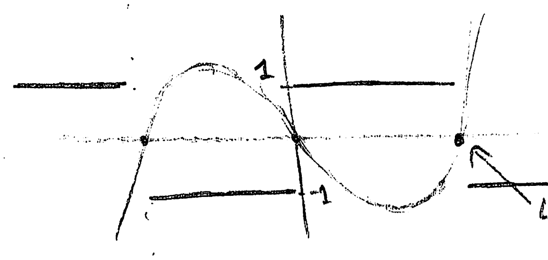
use the optimality conditions to derive the control law associated with the cost to go for this problem. Show your work.

Solution on next page

e) *Using the system and controller from part (d), what are the fixed points of the closed-loop system. For each fixed point, say if it is locally unstable, locally stable i.s.L, locally asymptotically stable, and/or locally exponentially stable.*

$$\hat{u} = \underset{u}{\text{argmin}} \left[x^2 + (4x^3 - 2x)(x+u) \right].$$

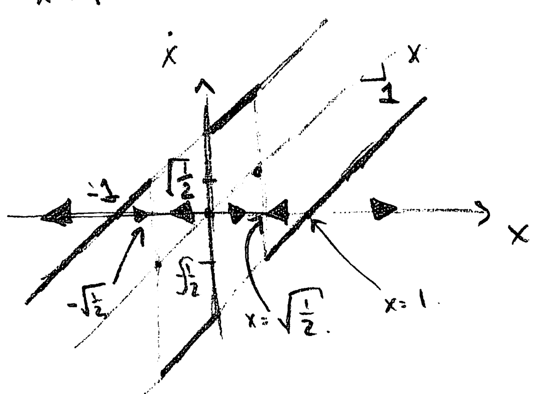
$$\hat{u} = -\text{sgn}(4x^3 - 2x). = \begin{cases} -1 & 4x^3 - 2x > 0 \\ 0 & 4x^3 - 2x = 0. \\ 1 & 4x^3 - 2x < 0. \end{cases}$$



$$4x^3 - 2x = 0 = x(2x^2 - 1) = 0$$

$$x = \pm\sqrt{\tfrac{1}{2}}, 0.$$

$$\dot{x} = \begin{cases} x - 1 & , \text{ for } 4x^3 - 2x > 0. \\ x & = 0 \\ x + 1 & > 0. \end{cases}$$



The closed-loop system has unstable fixed points at $x = 0$, $x = -1$, and $x = 1$.

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